

Generalized BPS magnetic monopoles

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We show the existence of Bogomol'nyi-Prasad-Sommerfield (BPS) magnetic monopoles in a generalized Yang-Mills-Higgs model which is controlled by two positive functions, $g(\phi^a\phi^a)$ and $f(\phi^a\phi^a)$. This effective model, in principle, would describe the dynamics of the nonabelian fields in a chromoelectric media. We check the consistency of our generalized construction by analyzing an explicit case ruled by a parameter β . We also use the well-known spherically symmetric *Ansatz* to attain the corresponding self-dual equations describing the topological solutions. The overall conclusion is that the new solutions behave around the canonical one, with smaller or greater characteristic length depending on the values of β .

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I. INTRODUCTION

In the context of classical field theories, configurations with nontrivial topology, generically named *topological defects*, arise as finite energy solutions to some nonlinear models. In the standard approach, such models are usually endowed by a symmetry breaking potential for the matter self-interaction, since topological defects are known to be formed during symmetry breaking phase transitions [1].

The simplest topological defect is the static kink, which arises in a (1+1)-dimensional theory containing a single real scalar field [2]. Other well-known examples of topological structures are the vortex configurations, which appear in effective planar models containing a complex scalar field coupled to an abelian gauge field [3], and the magnetic monopole, which stand for the static solution arising from a (1+3)-dimensional theory describing the interaction between a real scalar triplet and non-abelian gauge fields [4].

In particular, magnetic monopoles are spherically symmetric configurations coming from a static Yang-Mills theory endowed with a fourth-order Higgs potential [5]. In this case, these solutions exhibit no divergences and possess finite total energy, since they are constrained by a set of suitable boundary conditions. On the other hand, in the absence of a Higgs potential, magnetic monopoles arise as the minimal energy configurations of the corresponding Yang-Mills model [6]. In this context, such solutions come from a set of first-order differential equations, and minimize the total energy of the overall theory.

Furthermore, during the last years, some generalized or effective field theory models, generically named *k-field theories*, have been intensively investigated. The main difference between them and their canonical counterparts is the presence of nonstandard kinetic terms, which change the dynamics of the overall system in an exotic way. In fact, such theories have been used as effective models mainly in Cosmology, with the so-called *k-essence* models [7] suggesting new insights about the accelerated inflationary phase of the universe [8]. The

interesting point is that the introduction of generalized kinetic terms has important consequences on the formation of topological structures. Despite the possibility of achieving nontrivial topologically configurations even in the absence of the spontaneous symmetry breaking phenomenon [9], some *k-field* models endowed with spontaneous symmetry breaking also support topological defects [10]. In the last case, the resulting solutions can be studied via the comparison between them and their canonical counterparts, with the observation that they usually present slight variations on some of their main features [11].

On the other hand, in the presence of a generalized dynamics, the resulting model is highly nonlinear, its solutions being quite hard to find, even in the presence of suitable boundary conditions. In this context, the development of a consistent first-order framework is quite useful, since new topological configurations can be found by solving a set of generalized BPS equations. As it happens in the usual case, these new configurations minimize the energy of the overall system by saturating its lower bound [12].

Recently, some of us have performed the development of generalized first-order frameworks regarding several effective field theories [13]. However, for simplicity, these models were constructed containing only abelian fields. In the present work, we go further by presenting a first-order theoretical framework consistent with a generalized Yang-Mills-Higgs model. This model is controlled by two dimensionless positive functions, $g(\phi^a\phi^a)$ and $f(\phi^a\phi^a)$, which change the dynamics of the non-abelian fields in a nonusual way. Nevertheless, in order to guarantee the self-duality of the model, g and f are related by means of a simple constraint. As in the usual case, the total energy of the generalized model is also bounded from below, and the spherically symmetric self-dual solutions describe generalized BPS magnetic monopoles. Moreover, in the appropriate limit, our framework leads to the usual one, as expected.

In order to present our results, this work is outlined as follows. In Sec. II, we introduce the nonstandard Yang-Mills-Higgs model and the corresponding first-order the-

oretical framework, including its generalized BPS equations. Then, instead of recovering the usual case, we introduce an explicit generalized model, which is controlled by a single real parameter, β . In Sec. III, we present the numerical solution of the generalized BPS equations obtained by means of the relaxation technique. We depict the profiles of the Higgs and non-abelian gauge fields describing the generalized BPS magnetic monopoles. We also comment on the main features of these solutions. In Sec. IV, we finalize by summarizing our results and stating our perspectives.

II. THE THEORETICAL FRAMEWORK

We begin introducing a generalized version of the Yang-Mills-Higgs model which is defined by the following Lagrangian density:

$$\mathcal{L} = -\frac{g(\phi^a\phi^a)}{4}F_{\mu\nu}^bF^{\mu\nu,b} + \frac{f(\phi^a\phi^a)}{2}\mathcal{D}_\mu\phi^b\mathcal{D}^\mu\phi^b - V(\phi^a\phi^a), \quad (1)$$

where all fields, coordinates and parameters are supposed to be dimensionless (this can be achieved by using the appropriate mass rescaling transformations). We use standard conventions, including the natural units system. Here,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon^{abc}A_\mu^bA_\nu^c, \quad (2)$$

is the usual non-abelian field strength, and

$$\mathcal{D}_\mu\phi^a = \partial_\mu\phi^a + e\epsilon^{abc}A_\mu^b\phi^c, \quad (3)$$

stands for the non-abelian covariant derivative, $\eta^{\mu\nu} = (+ - - -)$ is the metric for the (1+3)-dimensional space-time, and ϵ^{abc} is the completely antisymmetric tensor (with $\epsilon^{123} = +1$). We point out that $g(\phi^a\phi^a)$ and $f(\phi^a\phi^a)$ are arbitrary functions which change the dynamics of the overall model in a nonusual way.

It is well-known that, in the usual case (i.e., given $g = f = 1$), self-dual configurations only exist in the absence of the Higgs potential. So, for simplicity, we also assume that their generalized counterparts only exist for $V(\phi^a\phi^a) = 0$. Furthermore, we suppose that such generalized configurations are described by the standard spherically symmetric *Ansatz*

$$\phi^a = \frac{x^a H(r)}{r} \quad \text{and} \quad A_i^a = \epsilon_{iak} \frac{x_k}{er^2} (W(r) - 1), \quad (4)$$

where $r^2 = x^a x_a$. Since we are searching for static configurations, we choose $A_0^a = 0$. The profile functions $H(r)$ and $W(r)$ are supposed to obey the usual finite energy boundary conditions

$$H(0) = 0 \quad \text{and} \quad W(0) = 1, \quad (5)$$

$$H(\infty) = \mp 1 \quad \text{and} \quad W(\infty) = 0. \quad (6)$$

These conditions guarantee not only the existence of finite energy configurations, but also the breaking of the $SO(3)$ symmetry inherent to the model (1).

In general, the weight functions $g(\phi^a\phi^a)$ and $f(\phi^a\phi^a)$ can be arbitrarily chosen, so that the Euler-Lagrange equations for $H(r)$ and $W(r)$ are quite hard to solve, even in the presence of the suitable boundary conditions (5) and (6). In this context, the development of a consistent self-dual theoretical framework is more than desirable, since it allows to find a set of first-order differential equations that leads to finite energy configurations attained by numerical procedures. These configurations are genuine solutions of the overall model, once they automatically fulfill the nonstandard Euler-Lagrange equations coming from Lagrangian (1).

From now on we focus our attention on the development of a BPS theoretical framework consistent with the nonstandard model (1). In order to perform it, we follow the usual approach, observing that the first-order equations come from the minimization of the total energy of the system, given by the energy-momentum tensor zero-zero component. In the present case, such tensor reads

$$T_{\lambda\rho} = f\mathcal{D}_\lambda\phi^a\mathcal{D}_\rho\phi^a - gF_{\mu\lambda}^aF^{a,\mu}_\rho - \eta_{\lambda\rho}\mathcal{L}, \quad (7)$$

from which one gets the nonstandard energy density (already written in terms of $H(r)$ and $W(r)$)

$$\begin{aligned} \varepsilon = & \frac{g}{e^2 r^2} \left(\left(\frac{dW}{dr} \right)^2 + \frac{(1 - W^2)^2}{2r^2} \right) \\ & + f \left(\frac{1}{2} \left(\frac{dH}{dr} \right)^2 + \left(\frac{HW}{r} \right)^2 \right), \end{aligned} \quad (8)$$

where $g = g(H)$ and $f = f(H)$. It is clear that such functions must be positive in order to avoid problems with the energy of the model.

Given the *Ansatz* (4), we point out that this model only engenders self-dual solutions when g and f are related to each other as

$$g(H) = \frac{1}{f(H)}. \quad (9)$$

In this case, the energy density (8) can be rewritten in the form

$$\begin{aligned} \varepsilon = & \frac{f}{2} \left(\frac{dH}{dr} \pm \frac{1 - W^2}{er^2 f} \right)^2 + \frac{1}{e^2 r^2 f} \left(\frac{dW}{dr} \mp efHW \right)^2 \\ & \mp \frac{1}{er^2} \frac{d}{dr} (H(1 - W^2)), \end{aligned} \quad (10)$$

whose minimization leads to the following first-order equations:

$$\frac{dH}{dr} = \mp \frac{1 - W^2}{er^2 f}, \quad (11)$$

$$\frac{dW}{dr} = \pm efHW. \quad (12)$$

Relations (11) and (12) are the self-dual (BPS) equations of the model (1). After implementing these equations, the BPS energy density becomes

$$\varepsilon_{bps} = \mp \frac{1}{er^2} \frac{d}{dr} (H (1 - W^2)), \quad (13)$$

and the total energy of the solutions is given by

$$E_{bps} = 4\pi \int r^2 \varepsilon_{bps} dr = \frac{4\pi}{e}, \quad (14)$$

whenever the boundary conditions (5) and (6) are fulfilled.

In summary, for a given positive function $f(H)$, the first-order equations (11) and (12) must be numerically solved in accordance with the finite energy boundary conditions (5) and (6). The self-dual solutions achieved in this way describe the generalized BPS magnetic monopoles arising from this nonstandard Yang-Mills-Higgs model (1), with total energy given by (14) and energy density by (13). However, it is worthwhile to point out that the generalized first-order framework presented in this letter only holds in the absence of the potential $V(\phi^a \phi^a)$. Furthermore, g and f have to obey the constraint (9). On the other hand, for a nonvanishing potential, or in the absence of the relation (9), such framework does not hold anymore, and the corresponding BPS monopoles can not be achieved.

The usual results are trivially recovered by setting $f = 1$. In order to show how this generalized framework works, we adopt the following generalization function:

$$f(H) = (H^2 + 1)^\beta, \quad (15)$$

where β is some real number; here, $\beta = 0$ leads us back to the canonical model. Now, given (15), the self-dual expressions (11) and (12) become

$$\frac{dH}{dr} = \mp \frac{1 - W^2}{er^2 (H^2 + 1)^\beta}, \quad (16)$$

$$\frac{dW}{dr} = \pm eHW (H^2 + 1)^\beta, \quad (17)$$

which must be solved respecting the conditions (5) and (6).

In the next Section, we solve the first order equations (16) and (17) by means of the relaxation technique for different values of β . Then, we plot not only the numerical results for $H(r)$ and $W(r)$, but also those for the BPS energy density (13), and for $r^2 \varepsilon_{bps}$ (the integrand of (14)). We also comment on the main features of the new solutions.

III. NUMERICAL RESULTS

Now, we focus our attention on the examination of the profiles of the generalized BPS solutions. Thus, we

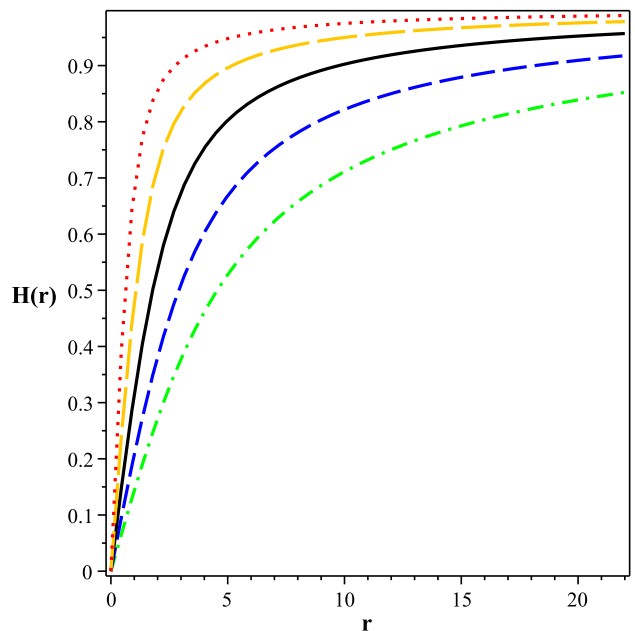


FIG. 1: Solutions to $H(r)$ for $\beta = -2$ (dot-dashed green line), $\beta = -1$ (dashed blue line), $\beta = 0$ (usual case, solid black line), $\beta = 1$ (long-dashed orange line) and $\beta = 2$ (dotted red line).

numerically solve the first-order equations (16) and (17) obeying the finite energy boundary conditions (5) and (6). Here, for simplicity, we choose $e = 1$, and consider only the lower sign in (6), (13), (16) and (17). Then, we numerically solve the self-dual system by means of the relaxation technique, for different values of the real parameter β .

Note that $\beta = 0$ leads us back to the usual model, whose self-dual solutions (already written according our conventions) can be attained analytically as

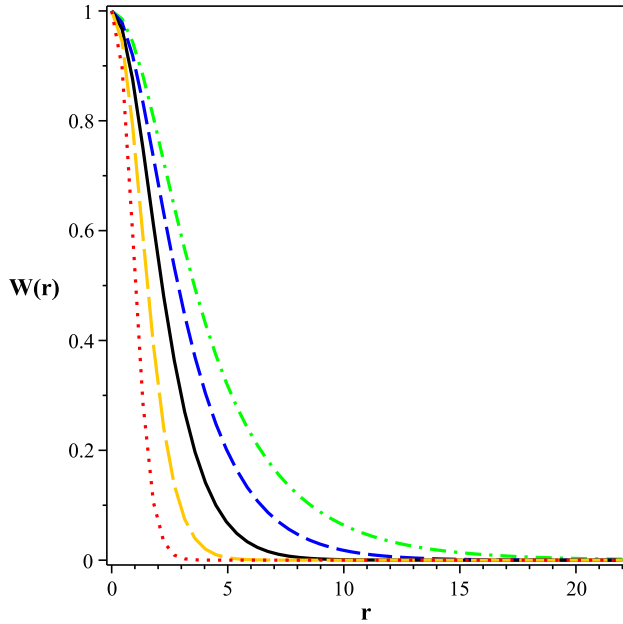
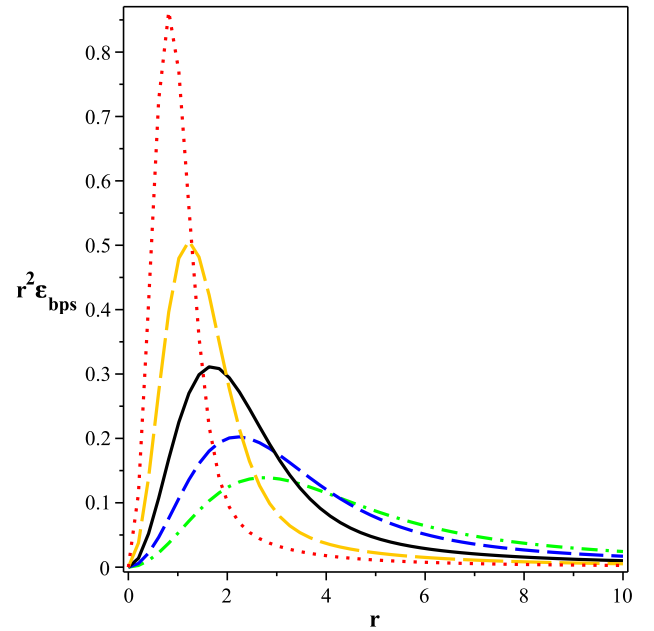
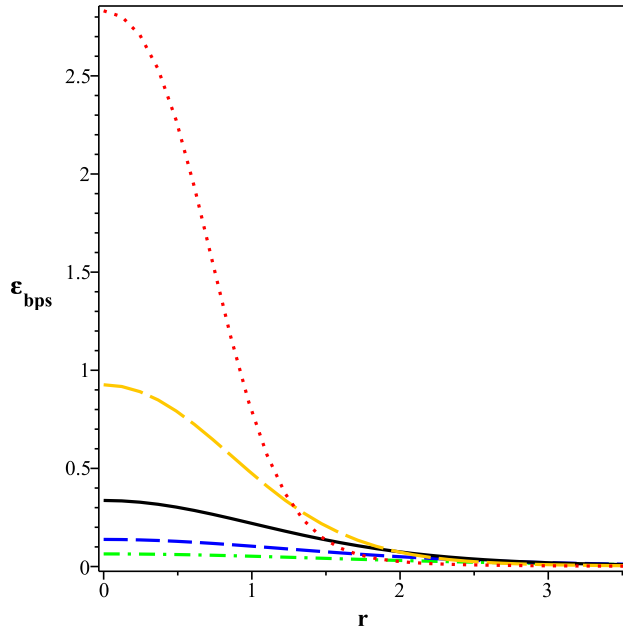
$$H(r) = \frac{1}{\tanh r} - \frac{1}{r}, \quad (18)$$

$$W(r) = \frac{r}{\sinh r}. \quad (19)$$

The numerical solutions we found for $H(r)$ and $W(r)$ are depicted in Figs. 1 and 2, for $\beta = -2$ (dot-dashed green line), $\beta = -1$ (dashed blue line), $\beta = 1$ (long-dashed orange line) and $\beta = 2$ (dotted red line). The usual (analytical) profiles are also shown (solid black line), for comparison. Moreover, we also plot the corresponding solutions for the BPS energy density (13) and for $r^2 \varepsilon_{bps}$ (the integrand of (14)) in Figs. 3 and 4.

In Figure 1, we present the numerical results regarding the profile of the function $H(r)$. In this case, we clearly see that the nonstandard solutions turn out depicted around the usual counterpart, behaving in the same general way, but changing the width of the defect.

We point out that different solutions engender different characteristic lengths. In particular, the solutions arising for $\beta < 0$ reach the asymptotic condition $H(\infty) = 1$

FIG. 2: Solutions to $W(r)$. Convention as in FIG. 1.FIG. 4: Solutions to $r^2 \varepsilon_{bps}$. Convention as in FIG. 1.FIG. 3: Solutions to ε_{bps} . Convention as in FIG. 1.

slower than the standard profile. In this sense, these solutions possess a greater characteristic length, so that the corresponding bosons mediate long-ranged interactions. On the other hand, the solutions related to $\beta > 0$ reach the saturation region faster, exhibiting smaller characteristic lengths, which is associated with small-ranged interactions.

In Fig. 2, we plot the solutions for $W(r)$. Again, the nonstandard solutions behave as the canonical one, exhibiting variations concerned with the defect width and

the characteristic length. Such variations are controlled by the real parameter β in the same way as before: for $\beta < 0$, the nonusual solutions engender the greater characteristic lengths, since they reach the asymptotic condition $W(\infty) = 0$ more slowly, being associated with long-ranged mediating bosons. Furthermore, for and increasing positive β , the solutions reach the saturation region faster, exhibiting smaller and smaller characteristic lengths, with the mediating bosons yielding small-ranged interactions.

The numerical profiles found for the BPS energy density ε_{bps} (13) are depicted in Fig. 3, revealing that such solutions are lumps centered at $r = 0$. Also, they vanish monotonically as r goes to ∞ (in fact, $\varepsilon_{bps}(r \rightarrow \infty) \rightarrow 0$ arises in a rather natural way from the asymptotic boundary conditions (6)). Here, the parameter β plays a different role, since it changes not only the characteristic lengths of the corresponding solutions, but also their amplitudes. In this sense, the solutions corresponding to $\beta < 0$ ($\beta > 0$) achieve the smaller (larger) amplitudes.

Finally, we depict the solutions for $r^2 \varepsilon_{bps}$, i.e., the integrand of (14); see Fig. 4. In this case, the solutions are rings centered at $r = 0$. Moreover, the points of larger amplitudes are located at some finite distance R from the origin (in this sense, R stands for the "radius" of the ring), such amplitudes being controlled by β in the same way as before. Here, we point out the existence of an interesting compensatory effect: the solutions reaching the greater amplitudes spread over smaller distances, and vice-versa. As a consequence, different solutions enclose the same area (equal to the unity, according our conventions), and the corresponding configurations achieve the same total energy; see (14).

IV. ENDING COMMENTS

In this work, we have presented a nonstandard first-order framework consistent with a generalized Yang-Mills-Higgs model (1), such model being controlled by two dimensionless functions, $g(\phi^a\phi^a)$ and $f(\phi^a\phi^a)$, which change the dynamics of the non-abelian fields in a non-usual way. In order to avoid problems with the energy of the system, these functions are supposed to be positive. We point out that the self-duality of the overall model only holds when g and f are related to each other by a simple constraint; see (9). Also, it is worthwhile to note that there is no additional constraint to be imposed on f .

The generalized first-order framework was developed in a general way. So, in order to verify the consistency of our construction, we have introduced an explicit example controlled by a single real parameter β ; see (15). Then, we have considered spherically symmetric self-dual configurations, the non-abelian fields being described by the standard static *Ansatz* (4). Moreover, the profile functions $H(r)$ and $W(r)$ were supposed to obey the usual finite energy boundary conditions given by (5) and (6).

The resulting first-order equations were solved numer-

ically by means of the relaxation technique, and the self-dual solutions we found were plotted in Figs. 1, 2, 3 and 4, for different values of β . The standard analytical solutions (achieved for $\beta = 0$) were also plotted, for comparison. The overall conclusion is that the nonstandard solutions behave in the same general way the usual one does, the main difference being slight variations on the amplitudes and on the characteristic lengths of the new solutions. In addition, we have identified the way such variations are controlled by the real parameter β .

Recently, some of us have performed detailed investigations addressing generalized self-dual frameworks for abelian models, attaining their respective first-order solutions [13]. This way, the present letter is an extension of those works to the non-abelian context. As for future investigations, interesting issues including the supersymmetric extension of the non-abelian model (1), and the search for its topological structures, are now under consideration.

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